ON THE PROBLEM OF STABILIZATION OF A NONLINEAR SYSTEM

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We consider a problem of constructing a control which would bring a nonlinear system to a given motion.

1. Let us describe the behavior of a system of automatic control by means of differential equations of perturbed motion

$$dx / dt = Ax + f(x) + bu \tag{1.1}$$

where x is an n-dimensional phase coordinate $\{x_i\}$ vector of the object; $u(t)$ is a scalar function describing the action of control; A in an n^2 matrix $\{a_{ij}\}$; b is an n-dimensional vector; $f(x)$ is an n-dimensional vector function of coordinates and $f(0) = 0$.

We assume that this function satisfies, over the whole phase space x, the Lipshits condition

$$| f_j(x)^{(2)} - f_j(x)^{(1)} | \leqslant L \| x^{(2)} - x^{(1)} \| \tag{1.2}$$

Here and in the following we use the notation

$$\| x \| = (x_1^2 + \ldots + x_n^2)^{1/2}$$

Let us consider the following problem. To find, for a given initial point x_0, a control $u(t)$ which would bring the system (1.1) to a given unperturbed motion $x = 0$ in time T.

Solution of this problem is not unique. We shall seek a control which, for small initial perturbations, is almost optimal provided that

$$J(u) = \max\left\{\max_\tau |u(\tau)|, \quad \nu \int_0^T |u(\tau)|\, d\tau\right\} = \min \qquad (\nu T > 1) \tag{1.3}$$

The problem of damping of a system of the first approximation

$$dx / dt = Ax + bu \tag{1.4}$$

under the condition that it is fully controllable [1] was considered, together with the character of the control (1.3), in [2 to 4]. In [5] an estimate was obtained of admissible initial deviations of the system (1.4), for which optimal control is possible. An estimate of its modulus with an upper bound H, is of the form

$$\| x \| \leqslant H\alpha / \sqrt{n} \qquad (\alpha = \text{const}) \tag{1.5}$$

2. Let us describe an iterative process of solving our problem. We consider the following Eq.:

$$x = x_0 + \int_0^T F(-\tau) f(x(\tau))\, d\tau \tag{2.1}$$

Here the integrand is calculated along a trajectory of (1.1) for the initial point x_0. The trajectory is generated by a control $u(\tau)$ which satisfies Eq.

$$0 = x + \int_0^T F(-\tau)\, bu(\tau)\, d\tau \tag{2.2}$$

Here $F(\tau)$ is the fundamental matrix of solutions of the homogeneous system (1.4).

We shall solve (2.1) using the method of consecutive approximations. Let us take $x = x_0$ and $u_{x0}(\tau) = u^\circ(\tau)$ where $u^\circ(\tau)$ is an optimal control of (1.4) (in the sense of (1.3)) calculated for x_0, as zero approximation. Control $u^\circ(\tau)$ satisfies (2.2). Eq. (2.1) yields

$$x_1 = x_0 + \int_0^T F(-\tau) f(x^\circ(\tau))\, d\tau \tag{2.3}$$

Control $u_{x_1}(t)$ is given by

$$u_{x_1}(\tau) = u_{x_0}(\tau) + \lambda_1(\tau) \tag{2.4}$$

where $\lambda_1(\tau)$ is the optimal control of the linear system (1.4) (in the sense of (1.3)) for the initial position $x(0) = x_1 - x_0$. At the same time, control (2.4) satisfies the relation

$$0 = x_1 + \int_0^T F(-\tau)\, b u_{x_1}(\tau)\, d\tau$$

Extending this process we find x_{n-1} and $u_{x_{n-1}}(\tau)$. Next approximation is then given by

$$x_n = x_0 + \int_0^T F(-\tau) f(x^{(n-1)}(\tau))\, d\tau, \qquad u_{x_n} = x_{x_{n-1}}(\tau) + \lambda_n(\tau) \tag{2.5}$$

where $\lambda_n(\tau)$ is the optimal control of (1.4) for the initial position $x(0) = x_n - x_{n-1}$. Also,

$$0 = x_n + \int_0^T F(-\tau)\, b u_{x_n}(\tau)\, d\tau \tag{2.6}$$

If the given iterative process converges, then Eq. (2.1) with the condition (2.2), has a solution x^* and $u^*(\tau)$. Combining the relations (2.1) and (2.2) written for these values we find that the control $u^*(\tau)$ takes the nonlinear system (1.1) from the point x_0 into the coordinate origin in time T.

3. In this section we shall discuss the conditions of convergence of the iterative process constructed in Section 2. The method used to construct successive approximations together with the inequality (1.5), lead to the following inequalities:

$$\int_0^T |u_{x_n} - u_{x_{n-1}}|\, d\tau \leqslant \gamma \|x_n - x_{n-1}\| \qquad (\gamma = \text{const}) \tag{3.1}$$

From (2.5) and (3.1), using the fundamental inequality of the pure theory of differential Eqs. in [6], we obtain

$$\|x_n - x_{n-1}\| \leqslant NL \|x_{n-1} - x_{n-2}\| \qquad (N = \text{const})$$

or

$$\|x_n - x_{n-1}\| \leqslant (NL)^{n-1} \|x_1 - x_0\| \tag{3.2}$$

Also, we have

$$\max_\tau |\lambda_n(\tau)| \leqslant \gamma_1 (NL)^{n-1} \|x_1 - x_0\| \quad (0 \leqslant \tau \leqslant T) \qquad (\gamma_1 = \text{const}) \tag{3.3}$$

Suppose now that

$$NL < 1 \tag{3.4}$$

holds. Then the iterative process constructed in Section 2, converges.

Indeed, in this case inequalities (3.2) and (3.3) guarantee the convergence of sequences of points $\{x_n\}$ and of functions $\{u_{x_n}\}$ $(0 \leqslant \tau \leqslant T)$. Moreover, sequence of functions $\{u_{x_n}\}$ converges uniformly, since, by definition, limit function is a sum of the following series:

$$u^*(\tau) = u^\circ(\tau) + \sum_{n=1}^{\infty} \lambda_n(\tau) \tag{3.5}$$

which, by (3.3), converges uniformly for $0 \leqslant \tau \leqslant T$. Let us denote the limit of the sequence $\{x_n\}$ by x^*. Vector x^* and function $u^*(\tau)$ satisfy (2.1) and (2.2). This follows from the fact that a limiting process is possible in relation (2.6) and from the uniform convergence of trajectories of the nonlinear system (1.1) originating at x_0 and generated by the sequence of controls $u_{x_n}(\tau)$ on the interval $[0, T]$.

Note. Since the inequality

$$\max_\tau | u_{x_n}(\tau) | \leqslant N_1 \| x_0 \qquad (N_1 = \text{const})$$

holds for any n, all trajectories of (1.1) taking part in the iterative process remain on the sphere

$$\|x\| \leqslant N_2 \| x_0 \| \qquad (N_2 = \text{const}) \tag{3.6}$$

Consequently, the requirement that condition (1.2) holds for the points of (3.6) only is sufficient for the iterative process to converge, provided (3.4), holds.

4. We shall assume that in sufficiently small neighborhood of the coordinate origin, nonlinearity of (1.1) represented by a vector function $f(x)$ satisfies an additional condition

$$| f_j(x) | \leqslant D \| x \|^{1+\varepsilon} \quad (j = 1, \ldots, n,\ D = \text{const}) \tag{4.1}$$

Following a procedure given in [7] we shall show that the constructed control $u^*(\tau)$ differs little from the optimal control of (1.1) in the sense of (1.3). Taking into account (1.2) and (4.1), we have

$$\| x_1 - x_0 \| \leqslant N_3 \| x_0 \|^{1+\varepsilon} \qquad (N_3 = \text{const}) \tag{4.2}$$

Let us denote the optimal control of (1.1) by $u_0(\tau)$ and apply it to the linear system (1.4), initial position of which is x_0. If $x(T)$ is the position of the linear system at the instant T, then

$$\| x(T) \| \leqslant N_4 \| x_0 \|^{1+\varepsilon} \qquad (N_4 = \text{const}) \tag{4.3}$$

Consider the control

$$u(\tau) = u_0(\tau) + \lambda(\tau) \tag{4.4}$$

where $\lambda(\tau)$ is the optimal control of (1.4) for the initial position $x(T)$. Also, by (1.5),

$$\max_\tau | \lambda(\tau) | \leqslant \gamma_1 \| x(T) \| \qquad (\gamma_1 = \text{const}) \tag{4.5}$$

Since the control (4.4) is admissible for a linear system, we have

$$\max_\tau | u^\circ(\tau) | - \max_\tau | \lambda(\tau) | \leqslant \max_\tau |u_0(\tau)| \leqslant \max_\tau | u^*(\tau) | \tag{4.6}$$

where $u^*(\tau)$ is a control of the nonlinear system (1.1), defined by (3.5).

Inequality (4.6) is written under the assumption that the minimum of (1.3) for systems (1.1) and (1.4) is determined by the first component. The argument that follows will be little changed if the value of the functional for the nonlinear system is determined by the second component, since components of (1.3) for the optimal control of a linear system are equal to each other.

From (4.6) together with (3.5), (3.3), (4.2), (4.3), (4.5) and the fact that $\max_\tau |u^\circ(\tau)|$ is of the order of $\|x_0\|$ we conclude that the values of $\max_\tau |u_0(\tau)|$ and $\max_\tau |u^*(\tau)|$ differ from each other by a magnitude of order not less than $\|x_0\|^{1+\varepsilon}$. This means that the value of (1.3) on the control $u_0(\tau)$ differs from the value of (1.3) on $u^*(\tau)$ by a magnitude of order not less than $\|x_0\|^{1+\varepsilon}$.

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